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A one-dimensional integrable model of fermions with multi-particle hopping*

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Abstract. A model with both single-particle and multi-particle hopping of electrons on a one-dimensional ‘triangular’ lattice is formulated and solved exactly by Bethe ansatz. On the basis of the exact calculation of the asymptotic behaviour of correlation functions we find a transition between the normal state and a state with a tendency to ‘superconductivity’. The latter state is confined to small densities of electrons up to a critical density ρ_c .

Low-dimensional electron systems with strong correlations are studied intensively because of their possible relevance to high- T_c superconductors. Although the main interest lies in the physics of two-dimensional systems, it is also extremely useful to investigate the one-dimensional variants of such models related to high- T_c superconductivity. A first reason is the conjecture that one- and two-dimensional models have many aspects in common [1]. Second, one-dimensional models are often integrable and have quite interesting physical behaviour. Moreover, such models also provide a testing ground for approaches intended for more complex problems. Examples are the one-dimensional Hubbard model [2], its supersymmetric extension [3], and the supersymmetric t - J model [4–6]. Recently we have found a few more integrable models with a tendency to superconductivity, namely the correlated hopping model [7, 8], an anisotropic t - J model [9], and a model with interchain tunnelling [10].

In this paper we propose a further new model which is exactly solvable and describes the competition between single-particle and multi-particle hopping in one dimension. The motivation for our investigation is the pair-hopping model proposed by Penson and Kolb [11]. In addition to the usual hopping term for electrons and an on-site repulsion energy U the Hamiltonian of this model contains a term for the hopping of singlet pairs of electrons from site to site:

$$\mathcal{H} = - \sum_j [c_{j1}^+ c_{j+1,1} + c_{j2}^+ c_{j+1,2} + V c_{j1}^+ c_{j,2}^+ c_{j+1,2} c_{j+1,1} + \text{HC}] + \sum_j U n_{j1} n_{j2}. \quad (1)$$

Here $c_{j\tau}^+$ creates an electron with spin index τ ($\tau = 1, 2$) on site j and $n_{j\tau} = c_{j\tau}^+ c_{j\tau}$ is the number operator, HC denotes the Hermitian conjugate.

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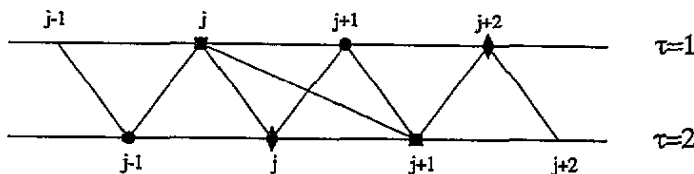


Figure 1. Depiction of the one-dimensional lattice showing the sites for electrons with spin $\tau = 1$ and 2 on an upper and a lower sublattice, respectively. The infinite Coulomb repulsion excludes the simultaneous occupation of nearest neighbours, i.e. the occupation of any triangle by more than one particle. The closest separation for two particles is realized by next-nearest neighbours represented by squares (disks or diamonds, respectively). The Hamiltonian consists of two competing interactions. The single-particle hopping describes the moving of individual particles on each sublattice by one lattice constant. The multi-particle term describes the hopping of a pair of particles on next-nearest neighbour sites to adjacent next-nearest neighbour sites, for instance from ‘square sites’ to ‘disk’ or ‘diamond’ sites.

This model is not exactly solvable and was studied numerically in the case of a half-filled band for $U = 0$ in [11] and for $U \neq 0$ in [12]. A further investigation was based on the renormalization group and bosonization method [13]. There are differences in the results of these different approaches and, in particular, the question about a possible phase transition and its location in the Penson–Kolb–Hubbard model remains open.

In this paper we consider a modified model which may be illustrated on a ‘one-dimensional triangular’ lattice (figure 1). Electrons with spin $\tau = 1$ move along the upper chain and electrons with spin $\tau = 2$ move along the lower chain. We assume that the Coulomb repulsion between electrons on nearest-neighbour sites is infinite which in the figure is illustrated by the ‘triangular structure’ of (infinite) repulsion. The explicit Hamiltonian of this model is given by

$$\mathcal{H} = - \sum_{j=1}^L \mathcal{P} \left[c_{j1}^+ c_{j+1,1} + c_{j2}^+ c_{j+1,2} + V c_{j1}^+ c_{j+1,2} (c_{j+1,1} c_{j-1,2} + c_{j+2,1} c_{j2}) + \text{HC} \right] \mathcal{P}. \quad (2)$$

The infinite Coulomb repulsion excludes the double occupancy of any two nearest neighbours on the lattice, and \mathcal{P} is the projector on the subspace of the allowed states. As in (1) the V -term describes pair hopping. L denotes the length of the chain and we assume periodic closure.

The advantage of this model (2) is its integrability, and the exact solution leads to interesting results as shown in the following. A simple comparison of Hamiltonians (1) and (2) shows that our model (2) keeps the main feature of the Penson–Kolb Hamiltonian, namely the competition between single-particle and pair hopping.

Hamiltonian (2) can even be generalized to an arbitrary number n of spin states, $\tau = 1, \dots, n$, while keeping integrability. In this case it contains additional multi-particle hopping terms and has the following form:

$$\mathcal{H} = - \sum_{j=1}^L \mathcal{P} \left[\sum_{\tau=1}^n c_{j\tau}^+ c_{j+1,\tau} + \sum_{l=1}^{n-1} V^l \sum_{\tau^{(1)} < \dots < \tau^{(l+1)}} \prod_{r=1}^{l+1} c_{j+r-1,\tau^{(r)}}^+ \times \left\{ \left(\prod_{s=1}^l c_{j+s,\tau^{(s)}} \right) c_{j-1,\tau^{(l+1)}} + c_{j+l+1,\tau^{(1)}} \left(\prod_{s=1}^l c_{j+s-1,\tau^{(s+1)}} \right) \right\} + \text{HC} \right] \mathcal{P}. \quad (3)$$

Here the infinite Coulomb repulsion excludes the occupancy of any sequence of $n+1$ nearest-neighbour sites, $(j, \tau), \dots, (j, n), (j + 1, 1), \dots, (j + 1, \tau)$, by more than one particle for

any τ .

Model (3) is the integrable generalization of (2). Therefore, we first present the solution for Hamiltonian (3), then the solution of model (2) is obtained as the most interesting special case, $n = 2$.

The exact solution for the eigenstates and eigenvalues of Hamiltonian (3) can be obtained within the framework of the Bethe ansatz method [14, 15, 4]. The central object of this method is the two-particle scattering matrix S which is calculated from the single- and two-particle processes described by Hamiltonian (3):

$$\begin{aligned}
 S_{\alpha\alpha}^{\alpha\alpha}(k) &= -e^{2ik} & S_{\alpha\beta}^{\alpha\beta}(k) &= \frac{\sin k}{\sin(k + i\eta)} e^{-2ik} \\
 S_{\alpha\beta}^{\beta\alpha}(k) &= -\frac{i \sinh \eta}{\sin(k + i\eta)} \exp[i\text{sign}(\beta - \alpha)k] e^{-2ik}
 \end{aligned}
 \tag{4}$$

where $k = k_1 - k_2$ and $\alpha \neq \beta = 1, \dots, n$. The system is integrable as the two-particle scattering matrix (4) satisfies the Yang-Baxter equation [14, 16]. Furthermore, there are no genuine N -particle scattering processes for $N \geq 3$ due to the particular arrangement of multi-particle interactions in (3). Therefore the mathematical conditions for eigenstates of (3) in the form of a Bethe ansatz are satisfied. For details of this method we refer the reader to the literature.

For finite systems suitable boundary conditions have to be imposed, e.g. periodic ones. In this case the momentum k_j of each particle is subject to certain quantization conditions involving the total scattering phase upon the other particles. This leads to the diagonalization problem of products of scattering matrices solved by a subsequent Bethe ansatz. The result is given in terms of spin rapidities $\Lambda_\alpha^{(r)}$ and nested Bethe ansatz equations

$$\begin{aligned}
 \exp(ik_j L) &= \prod_{i=1}^N \exp[i(k_j - k_i)] \prod_{\alpha=1}^{M_{n-1}} \frac{\sin[\frac{1}{2}(k_j - \Lambda_\alpha^{(1)} + i\eta/2)]}{\sin[\frac{1}{2}(k_j - \Lambda_\alpha^{(1)} - i\eta/2)]} & j = 1, \dots, N \\
 \prod_{\beta=1}^{M_{n-r}} \frac{\sin[\frac{1}{2}(\Lambda_\alpha^{(r)} - \Lambda_\beta^{(r)} + i\eta)]}{\sin[\frac{1}{2}(\Lambda_\alpha^{(r)} - \Lambda_\beta^{(r)} - i\eta)]} &= - \prod_{\beta=1}^{M_{n-r+1}} \frac{\sin[\frac{1}{2}(\Lambda_\alpha^{(r)} - \Lambda_\beta^{(r-1)} + i\eta/2)]}{\sin[\frac{1}{2}(\Lambda_\alpha^{(r)} - \Lambda_\beta^{(r-1)} - i\eta/2)]} \\
 &\times \prod_{\beta=1}^{M_{n-r-1}} \frac{\sin[\frac{1}{2}(\Lambda_\alpha^{(r)} - \Lambda_\beta^{(r+1)} + i\eta/2)]}{\sin[\frac{1}{2}(\Lambda_\alpha^{(r)} - \Lambda_\beta^{(r+1)} - i\eta/2)]} & r = 1, 2, \dots, n-1.
 \end{aligned}
 \tag{5}$$

where N is the number of particles. We have introduced the interaction parameter η ($<$ or $>$ 0) via $V = e^{-\eta}$. Furthermore we denote the number of particles with spin index τ by m_τ such that

$$M_l = \sum_{\tau=1}^l m_\tau \tag{6}$$

is the total number of particles with spin indices $1, 2, \dots, l$. Thus we have $M_n = N$, $M_0 = 0$ and $\Lambda_j^{(0)} = k_j$.

The total energy and momentum of the system are given in terms of the electron rapidities k_j as

$$E = -2 \sum_{j=1}^N \cos k_j \quad P = \sum_{j=1}^N k_j. \tag{7}$$

Equations (5) hold regardless of the sign of η , nevertheless the structure of the solutions is very different for $\eta > 0$ and $\eta < 0$. In the case $\eta > 0$ ($V < 1$) there are no complex roots

for the k_j . It can be expected that single-particle hopping dominates in (2), (3) and that the particles move independently and there are no bound (Cooper) pairs in the system. In the absence of an external field the ground state corresponds to a symmetric configuration with the same number of particles $m = m_\tau$ for all spin states $\tau = 1, 2, \dots, n$. Equations (5) are straightforwardly [7] reduced to one set of equations for real k_j , which in the thermodynamic limit ($L \rightarrow \infty$) leads to an integral equation for the density distribution $\rho(k)$:

$$2\pi\rho(k) - \int_{-k_0}^{k_0} \varphi(k-k')\rho(k') dk' = 1$$

$$\varphi(k) = -\frac{1}{n} + 2 \sum_{\nu=1}^{\infty} \exp(-\nu\eta) \frac{\sinh[\nu\eta(n-1)]}{\sinh(\nu\eta n)} \cos(\nu k) \quad (8)$$

where k_0 is determined by the subsidiary condition for the particle density ρ ,

$$\int_{-k_0}^{k_0} \rho(k) dk = N/L = \rho. \quad (9)$$

The energy is given by

$$\frac{E}{L} = -2 \int_{-k_0}^{k_0} \cos k \rho(k) dk. \quad (10)$$

In the opposite case $\eta < 0$ ($V > 1$) one expects that the pair-hopping term in (2) and, more generally, the multi-particle-hopping terms in (3) dominate over single-particle hopping and might lead to a coherent motion of n particles, one from each spin state, and ultimately to bound n -particle complexes. This is indeed the case. In the ground state the tendency to form bound complexes is maximal and is reflected in the distribution of $\Lambda_\alpha^{(r)}$ values. A complex of n particles is described by one real $\Lambda^{(n-1)}$ value and associated Λ values given by [15, 17]

$$\Lambda^{(r,p)} = \Lambda^{(n-1)} + i\frac{p}{2}\eta \quad p = -(n-r-1), -(n-r-3), \dots, (n-r-1)$$

$$r = 0, 1, 2, \dots, n-1 \quad (11)$$

up to corrections which are exponentially small in the thermodynamic limit. There are M complexes such that $N = Mn$. Using (11) the Bethe ansatz equations (5) are reduced to one set of M equations for the real $\Lambda_\alpha = \Lambda_\alpha^{(n-1)}$:

$$nL\Lambda_\alpha - n^2 \sum_{\beta=1}^M (\Lambda_\alpha - \Lambda_\beta) = 2\pi J_\alpha + \sum_{\beta=1}^M \sum_{l=1}^{n-1} \Theta(\Lambda_\alpha - \Lambda_\beta; l\eta) \quad \alpha = 1, \dots, M$$

$$\Theta(\Lambda; \eta) = 2 \arctan(\coth \eta \tan \Lambda/2) \quad -\pi < \Theta(\Lambda; \eta) \leq \pi. \quad (12)$$

The energy is given by

$$E = \sum_{\alpha=1}^M \epsilon_0(\Lambda_\alpha) \quad \epsilon_0(\Lambda) = -2 \frac{\sinh(n\eta)}{\sinh \eta} \cos \Lambda. \quad (13)$$

The J_α in (12) are integer (half-integer) numbers for even (odd) $(n-1)(M-1)$.

In the thermodynamic limit, $L, M \rightarrow \infty$, equation (12) leads to an integral equation for the distribution function $\sigma(\Lambda)$ of particle complexes:

$$2\pi\sigma(\Lambda) + \int_{-\Lambda_0}^{\Lambda_0} \left[\sum_{l=1}^{n-1} \Theta'(\Lambda - \tilde{\Lambda}; l\eta) \right] \sigma(\tilde{\Lambda}) d\tilde{\Lambda} = n(1 - \rho) \quad (14)$$

with subsidiary condition

$$\int_{-\Lambda_0}^{\Lambda_0} \sigma(\Lambda) d\Lambda = \frac{M}{L} = \frac{\rho}{n}. \tag{15}$$

In order to study the superconducting properties of the model we calculate the long-distance behaviour of correlation functions. In the Bethe ansatz approach it is a formidable task to deal with correlation functions. However, due to developments in two-dimensional conformal field theory the scaling dimensions describing the algebraic decay of correlations are, in principle, accessible [18, 19]. According to this theory there is a one-to-one correspondence between the conformal dimensions of the scaling operators and the finite-size corrections for the energies of the excited states of the critical Hamiltonian. Using this method we have studied critical properties of an n -sublattice fermion model with correlated hopping [7]. Fortunately, the Bethe ansatz equations for our present model have the same structure as the correlated hopping model and can be analysed in the same way. Therefore, omitting the details of the calculation, we only present the results.

Our model has only one type of gapless excitation. These excitations correspond to the motion of particles and particle complexes for $\eta > 0$ and $\eta < 0$, respectively. As in [7] the scaling dimensions of the different fields are uniquely expressed in terms of the dressed charge Z [19, 20]:

$$x = (\Delta\mathcal{N}/2Z)^2 + Z^2 d^2 + N^+ + N^- \tag{16}$$

where $\Delta\mathcal{N}$, N^+ , N^- and d label the quantum numbers which specify gapless excitations. $\Delta\mathcal{N} = \Delta N$ (ΔM) is the change in the number of particles (complexes of particles) as compared to the ground state, N^+ , N^- are non-negative integers and d is the number of particles (complexes of particles) excited from the left Fermi point to the right one. Z is the dressed charge at the Fermi point

$$Z = \begin{cases} \xi(k_0) & \eta > 0 \\ \zeta(\Lambda_0) & \eta < 0 \end{cases} \tag{17}$$

where the integral equations for the functions ξ and ζ can be obtained from equations (8) and (14), respectively. As a result we have

$$\xi(k) = 2\pi\rho(k) \quad \zeta(\Lambda) = 2\pi\sigma(\Lambda)/n. \tag{18}$$

The static two-point correlation functions of a scaling field $\Phi(r)$ at site r with conformal dimension x are given by

$$\langle \Phi(r)\Phi(0) \rangle = \exp(-2idk_{\text{F}}r)/r^{2x} \tag{19}$$

where

$$k_{\text{F}} = \begin{cases} \pi\rho & \eta > 0 \\ \pi\rho/n & \eta < 0. \end{cases} \tag{20}$$

Let us first consider the case $\eta > 0$. The leading contribution to the long-distance power-law behaviour of the density-density correlation function is given by $d = 1/n$, $N^{\pm} = 0$ such that

$$\langle \rho(r)\rho(0) \rangle \simeq \rho^2 + Ar^{-\alpha} \cos(2k_{\text{F}}r) \tag{21}$$

where

$$\rho(r) = \sum_{\tau=1}^n c_{j\tau}^+ c_{j\tau} \quad \alpha = 2[\xi(k_0)]^2/n^2 \tag{22}$$

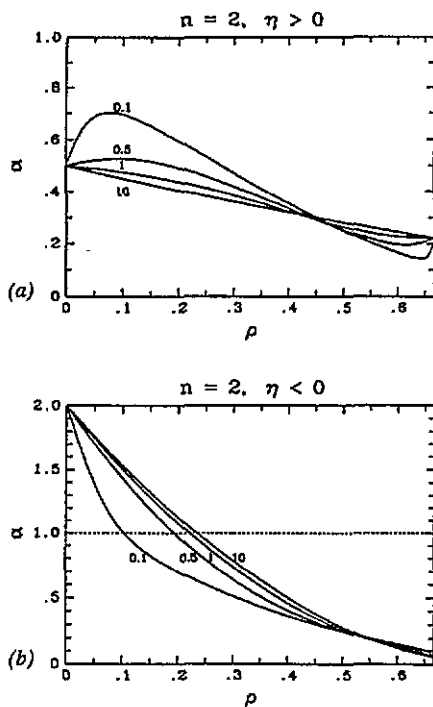


Figure 2. (a) The critical exponent α in dependence on the particle density ρ for $n = 2$ and different values of $\eta = 0.1, 0.5, 1, 10$. (b) The same as for (a) with negative values of η , $|\eta| = 0.1, 0.5, 1, 10$. Within graphical resolution the curves for $\eta = 10$ cannot be distinguished from $\eta = \infty$.

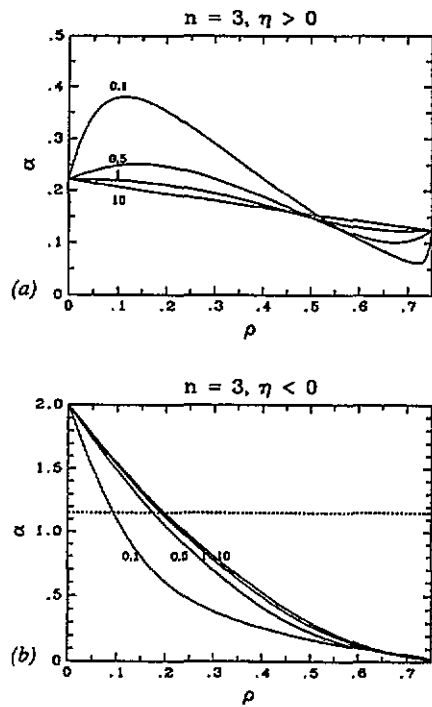


Figure 3. The critical exponent α for $n = 3$, similar to figure 1.

and α is the critical exponent. To consider the superconducting properties of the system we discuss the correlations of complexes of particles, created by

$$C^+(r) = \prod_{\tau=1}^n c_{r,\tau}^+ \quad (23)$$

In the simplest case $n = 2$ this corresponds to the correlations of singlet pairs. The asymptotic behaviour of the correlation function is given by $\Delta\mathcal{N} = n$ and $d = 0$ or $1/2n$ for even or odd n , respectively. Then we obtain

$$\langle C^+(r)C(0) \rangle \simeq Br^{-\beta} \quad \beta = \alpha^{-1} \quad (24)$$

for n even and

$$\langle C^+(r)C(0) \rangle \simeq Br^{-\beta} \cos(k_{\text{F}}r) \quad \beta = \alpha^{-1} + \alpha/4 \quad (25)$$

for n odd, where again β is the critical exponent.

Similarly we consider the case $\eta < 0$. The expressions (21), (24), (25) also hold in this case, but the critical exponent α is now given in terms of $\zeta(\Lambda_0)$ as

$$\alpha = 2[\zeta(\Lambda_0)]^2 \quad (26)$$

The equations (21)–(26) have a universal structure which is characteristic for a critical model with one type of gapless excitation. The critical exponents (21)–(25) can be obtained

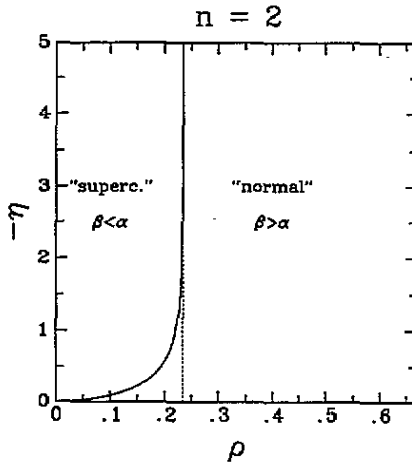


Figure 4. The phase diagram for $\eta < 0$ and $n = 2$. The full line separates regions with dominating 'superconducting' correlations and dominating density-density correlations ($\beta < \alpha$ and $\beta > \alpha$), respectively.

from the critical exponents of the correlated hopping model [7] by changing the kernel of the integral equation (8) $\varphi(k)$ to $\bar{\varphi}(k)$, which are simply related by

$$\varphi(k) = \bar{\varphi}(k) - 1. \tag{27}$$

The critical exponent of our present model $\alpha(\rho, \eta)$ is expressed in terms of the critical exponent of the correlated hopping model $\alpha_{(c)}(\rho, \eta)$ as

$$\alpha(\rho, \eta) = (1 - \rho)^2 \alpha_{(c)}\left(\frac{\rho}{(1 - \rho)}, \eta\right) \quad \text{for } \eta > 0 \tag{28}$$

and

$$\alpha(\rho, \eta) = (1 - \rho)^2 \alpha_{(c)}\left(n - \frac{\rho}{(1 - \rho)}, |\eta|\right) \quad \text{for } \eta < 0. \tag{29}$$

In both cases the density ρ ranges in

$$0 \leq \rho \leq \rho_{\max} = n/(n + 1) \tag{30}$$

where the maximal density ρ_{\max} is determined from the exclusion of site occupancies (see figure 1).

The solution for $\alpha_{(c)}(\rho, \eta)$ has been studied numerically in [7]. For $\eta > 0$ this function has the following behaviour. It varies monotonically from $\alpha_{(c)}(\rho = 0) = 2/n^2$ to $\alpha_{(c)}(\rho = n) = 2$. This shows that, in contrast to the correlated hopping model, in our present model for all n there are no densities ρ for which $\beta < \alpha$ if $\eta > 0$, cf figures 2(a) and 3(a). (Note that $\alpha(\rho)$ is a non-monotonic function.) In this case single-particle motion dominates over the motion of multi-particle complexes. Therefore there is no tendency towards superconductivity for any particle density.

For $\eta < 0$ the situation is quite different. For all n there exists a critical density ρ_c and an interval $[0, \rho_c]$ for which $\beta < \alpha$, see figures 2(b) and 3(b). In this case the correlation function of n -particle complexes has a slower decay than the density correlation function and thus dominates. This indicates the tendency to 'superconductivity'. In figure 4 we show the phase diagram with the transition line from normal to 'superconducting' behaviour. We

suppose that such kind of a transition also exists in the original Penson–Kolb–Hubbard model (2) as suggested by numerical calculations in [11].

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References

- [1] Anderson P W 1990 *Phys. Rev. Lett.* **64** 1839
- [2] Lieb E H and Wu F Yu 1968 *Phys. Rev. Lett.* **20** 1445
- [3] Essler F H L, Korepin V E and Schoutens K 1992 *Phys. Rev. Lett.* **68** 2960; 1993 *Phys. Rev. Lett.* **70** 73
- [4] Sutherland B 1975 *Phys. Rev. B* **12** 3795
- [5] Schlottmann P 1987 *Phys. Rev. B* **36** 5177
- [6] Bares P A and Blatter G 1990 *Phys. Rev. Lett.* **64** 2567
- [7] Bariev R Z, Klümper A, Schadschneider A and Zittartz J 1993 *J. Phys. A: Math. Gen.* **26** 1249, 4863
- [8] Zittartz J, Klümper A, Schadschneider A and Bariev R Z 1994 *Physica B* **194–196** 1417
- [9] Bariev R Z, Klümper A, Schadschneider A and Zittartz J 1995 *Z. Phys. B* **96** 395
- [10] Bariev R Z, Klümper A, Schadschneider A and Zittartz J 1994 *Phys. Rev. B* **50** 9676
- [11] Penson K and Kolb M 1986 *Phys. Rev. B* **33** 1663
Kolb M and Penson K 1986 *J. Stat. Phys.* **44** 129
- [12] Hui A and Doniach S 1993 *Phys. Rev.* **48** 2063
- [13] Affleck I and Marston J B 1988 *J. Phys. C: Solid State Phys.* **21** 2511
- [14] Yang C N 1967 *Phys. Rev. Lett.* **19** 1312
- [15] Gaudin M 1983 *La Fonction d'Onde de Bethe* (Paris: Masson)
- [16] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
- [17] Takahashi M 1971 *Prog. Theor. Phys.* **46** 401
- [18] Belavin A A, Polyakov A M and Zamolodchikov A B 1984 *Nucl. Phys. B* **241** 333
- [19] Cardy J L 1986 *Nucl. Phys. B* **270** 186
- [20] Izergin A G, Korepin V E and Reshetikhin N Yu 1989 *J. Phys. A: Math. Gen.* **22** 2615